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Thermodynamics Properties of the Inner Horizon of a Kerr-Newman Black Hole

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Abstract In this paper, we study the thermal properties of the inner horizon of a Kerr-Newman black hole. By adopting Damour-Ruffini method and the thin film model which is developed on the base of brick wall model suggested by 't Hooft, we calculate the temperature and the entropy of the inner horizon of a Kerr-Newman black hole. We conclude that the temperature of inner horizon is positive and the entropy of the inner horizon is proportional to the area of the inner horizon. The cut-off factor is same as it in calculation of the entropy of the outer horizon, 90β. In addition, we write the integral and differential Bekenstein-Smarr formula as the parameters of the inner horizon. Then, we discuss that if the contribution of the inner horizon is taken into account to the total entropy of the black hole, the Nernst theorem can be satisfied. At last, We calculate the tunneling rate of the outer horizon Γ_+ and the inner horizon Γ_-. The total tunneling rate Γ should be the product of the rates of the outer and inner horizon, $\Gamma = \Gamma_+ \cdot \Gamma_-$. We find that the total tunneling rate is in agreement with the Parikh's standard result, $\Gamma \to \exp(\Delta S_{BH})$, and there is no information loss.

Keywords Inner horizon · Tunneling effect · Nernst theorem

1 Introduction

Bekenstein first suggested that the entropy of a black hole is proportional to its surface area [1]. Then, Hawking made a striking discovery that basic principles of quantum field theory lead to the emission of thermal radiation from a classical black hole [2]. However, our understanding of the possible role played by the inner horizon in black hole thermodynamics is still somewhat incomplete.

To the K-N space time, the Killing vector field $(\frac{\partial}{\partial t})^a$ both outside the outer horizon and inside the inner horizon is time-like, where the space-time is stationary. So, the surface gravity on the inner horizon can be well defined. Using the formula in the [3], we get the

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result that κ_- is negative, unlike the surface gravity $\kappa_+ > 0$ on the outer horizon. This can be understood easily. The outer horizon is a future horizon to an observer outside it. The matter can only fall into the black hole and can not escape from it in a classical situation. However, the inner horizon is a past horizon to a observer inside it. The observer inside the inner horizon regards it as a horizon of a white hole. The white hole can emit matter classically, not absorb matter. So, the surface gravity of outer horizon, κ_+, is directed to the horizon and is positive; κ_-, is directed to the singularity, not to the inner horizon and it is reasonable that it is negative. Hence, the tunneling process near the inner horizon is that the particle tunnels from $r < r_-$ to $r > r_-$. Using Damour-Ruffini method [4], we prove that the temperature of the inner horizon is also positive and is a constant all over the inner horizon. We can explain the Hawking radiation of the K-N black holes as follows. A flow of positive energy particles produced near the singularity propagates in time and reach the inner horizon. Then, these particles are scattered by the inner horizon and travel in the reversed time toward the outer horizon where they are scattered again. At last, they travel forward in time to infinity appearing as Hawking radiation.

We calculate the entropy of the inner horizon by using the thin film brick wall model [5] which is on the base of brick wall model proposed by 't Hooft [6]. Because the entropy is associated with the field in the small region where exists the local thermal equilibrium and the statistical laws are valid [7]. So, the field outside the outer horizon can be supposed to non-zero only in the thin film $(r_+ + \varepsilon \to r_+ + \varepsilon + \delta)$, where r_+ is the radius of the outer horizon, ε is the ultraviolet cut-off and δ is the thickness of the thin film. Using this model we can work out the entropy of the outer horizon. There exists a time-like Killing vector field in the region $r < r_-$. The field in the thin film $(r_- - \varepsilon \to r_- - \varepsilon - \delta)$ can be regarded as non-zero when we calculate the entropy of the inner horizon. We obtain that the entropy of the inner horizon is also proportional to its area and the cut-off is 90β.

There is still an open problem of the entropy of the black hole [8–10]. According to Nernst theory of the third law of ordinary thermodynamics, the entropy of a system must go to zero as its temperature reaches zero. If this assertion is used to black holes, we find that the entropy of the black hole with two horizons, such as Kerr black hole, does not go to zero as its temperature approaches absolute zero [11, 12]. If the black hole with two horizons is regarded as a thermodynamics system composed of two subsystem, the outer horizon and inner horizon, the entropy of the black hole should include the contribution both of the outer and inner horizon [13]. In this paper, as an example of the black holes with two horizon, we propose that the entropy of the K-N black hole can be written as:

$$S_{BH} = S_+ + S_-, \tag{1}$$

where, S_+ and S_- are the entropy contributed by the outer and inner horizon respectively. When the temperature of the K-N black hole approaches zero, the total redefined entropy, $S_{BH} = S_+ + S_-$, vanishes. The Nernst theorem is satisfied.

Recently, Parikh and Wilczek gave an enlightening suggestion that Hawking radiation could be treated as a tunneling precess [14–16]. The self-interaction effect is taken into account in their method. They obtained a leading correction to the emission rate arising from loss of mass of the black hole and concluded that the information was conserved. Following this method, Zhang and Zhao have extended Parikh's method from static black holes to the non-spherical symmetric stationary black holes and radiation of charged particle and massive particle [17–20]. However, the information is not conserved if we only consider the tunneling process of the outer horizon after redefining the entropy of the K-N black hole. In this paper, we have a new idea that the tunneling effect of the inner horizon must be taken into account because there do exist thermal characters of the inner horizon. A positive energy particle created near the singularity travels forward in time and arrives at the inner horizon where the tunneling rate is Γ_-. Then, it goes in the reversed time toward the outer horizon. The tunneling rate is Γ_+ at the outer horizon. The total tunneling rate Γ should be the product of the rates of the outer and inner horizon, $\Gamma = \Gamma_+\Gamma_-$. Our result is in agreement with the Parikh's standard result, $\Gamma \to \exp(\Delta S_{BH})$, and the information is conserved, where S_{BH} is the sum of the contribution of the outer and inner horizon.

The paper is organized as follows. In Sects. 2 and 3 we calculated the temperature and the entropy of the inner horizon of the K-N black hole respectively. In Sect. 4, we discuss that the Bekenstein-Smarr formula can be written as the parameters of the inner horizon. In Sect. 5, we study the tunneling effect of two horizons. Throughout the paper, the units $G = c = \hbar = k_B = 1$ are used.

2 Temperature of the Inner Horizon of the K-N Black Hole

The line element of Kerr-Newman black hole is described by

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{(2Mr - Q^2)a^2\sin^4\theta}{\rho^2}\right]d\varphi^2 - \frac{2(2Mr - Q^2)a\sin^2\theta}{\rho^2}dt_s d\varphi. \quad (2)$$

The radiuses of the outer horizon and the inner horizon satisfy

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \quad (3)$$

The angular velocity and the surface gravity of the outer and inner horizon are

$$\Omega_\pm = \frac{a}{r_\pm^2 + a^2}, \quad (4)$$

$$\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)}, \qquad \kappa_- = -\frac{r_+ - r_-}{2(r_-^2 + a^2)} \quad (5)$$

Note that κ_- is negative. It is because that the inner horizon is a horizon of white hole and the surface gravity of the inner horizon is directed to the singularity, not to the horizon, unlike κ_+ which is directed to the outer horizon.

In 1976, Damour and Ruffini suggested a method which can demonstrate the Hawking radiation [4]. The Klein-Gordon equation is

$$(\Box - \mu^2)\Phi = 0. \quad (6)$$

Making the separation of variables as, $\Phi = e^{im\varphi - i\omega t}\psi(r)\Theta(\theta)$, the radical equation can be written as

$$\Delta\frac{d^2\psi}{dr^2} + 2(r - M)\frac{d\psi}{dr} = \left(\lambda + \mu^2 r^2 - \frac{K^2}{\Delta}\right)\psi, \quad (7)$$

where $K = (r^2 + a^2)\omega - am$. Introducing the tortoise coordinate transformation:

$$r_* = r + \frac{1}{2\kappa_+} \ln\left(\frac{|r - r_+|}{r_+}\right) - \frac{1}{2|\kappa_-|} \ln\left(\frac{|r - r_-|}{r_-}\right), \tag{8}$$

when, $r \to r_\pm$, $\Delta \to 0$, the (7) can be written as [13]

$$\frac{d^2\psi}{dr_*^2} + (\omega - \omega_0)^2\psi = 0, \tag{9}$$

where $\omega_0 = m\Omega_\pm$. Equation (9) shows that there are waves which propagate radically near the outer and inner horizon. It is well known that Hawking radiation exists near the outer horizon. In this paper, we are only interested in the case near the inner horizon ($r < r_-$). Here ω_0 is adopted by $m\Omega_-$. Because the inner horizon can be regarded as a past horizon, we adopt the retarded Eddington-Finkelstein coordinate $u = t - [(\omega - \omega_0)/\omega]r_*$ [13]. The solution near the inner horizon is

$$\phi_{in} = e^{-i\omega t - i(\omega - \omega_0)r_*} = e^{-i\omega u - 2i(\omega - \omega_0)r_*}. \tag{10}$$

As $r \to r_-$,

$$r_* \to \frac{1}{2\kappa_-} \ln(r_- - r). \tag{11}$$

Thus, the ingoing wave can be written as

$$\psi_{in} = e^{-i\omega u}(r_- - r)^{-\frac{i(\omega - \omega_0)}{\kappa_-}}. \tag{12}$$

We can find that ψ_{in} is not analytic at the inner horizon. Using the analytical extension [22, 23], the thermal spectrum and temperature of the inner horizon are, respectively,

$$N_\omega^2 = \frac{1}{e^{\frac{\omega - \omega_0}{T_-}} \pm 1}, \quad T_- = \frac{-\kappa_-}{2\pi}. \tag{13}$$

The temperature is a constant all over the inner horizon. The particle tunnels from inside the inner horizon into the one-way membrane region because the inner horizon is a white hole horizon. Therefore we have proved that there does exist radiation from the region $r < r_-$ to the inner horizon. The effect can be regarded as "Hawking absorption" [22]. The outer horizon is in thermal equilibrium with the thermal radiation outside the black hole. Similarly, the inner horizon is in thermal equilibrium with the thermal radiation inside the inner horizon. The inner horizon absorbs thermal radiation at temperature T_-, and at the same time it emits thermal radiation at temperature T_-. So, the inner horizon is a thermal system with temperature T_-. We can explain Hawking radiation as follows. The positive energy particles created near the singularity are scattered by the inner horizon. Then, they travel in the reverse time direction, transiting the "one-way membrane" region and arrive at the outer horizon. After scattered by the out horizon, they travel to infinity as Hawking radiation.

3 Entropy of the Inner Horizon of the K-N Black Hole

We use the thin film brick model [5, 7, 24] to calculate the entropy of the inner horizon. Let us substitute the metric into Klein-Gordon equation, which describes the scalar field with mass μ:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\right)-\mu^2\Phi=0. \tag{14}$$

The solution can be written as

$$\Phi=e^{-i\omega t+im\varphi}R(r)\Theta(\theta). \tag{15}$$

With the WKB approximation, we get the wave vector:

$$k_r=\frac{r^2+a^2}{\Delta}\sqrt{\left(\omega-\frac{a}{r^2+a^2}m\right)^2-\frac{\Delta}{(r^2+a^2)^2}(\lambda+\rho^2\mu^2+\aleph)}, \tag{16}$$

where $\aleph=a^2\sin^2\theta\omega^2+\frac{m^2}{\sin^2\theta}-2am\omega$. According to quantum statistical mechanics, the free energy is given by

$$F=-\frac{1}{\pi}\int_0^\infty d\widetilde{\omega}\int_{r_--\varepsilon}^{r_--\varepsilon-\delta}dr\int\frac{k_r}{e^{\beta\widetilde{\omega}}-1}d\lambda, \tag{17}$$

where $\widetilde{\omega}=\omega-\Omega_-m$. Studying the integration on λ,ω, we get

$$F=-\frac{2\pi^3}{45\beta^4}\int_{r_--\varepsilon}^{r_--\varepsilon-\delta}\frac{(r^2+a^2)^3}{(r-r_+)^2(r-r_-)^2}dr \tag{18}$$

$$=-\frac{2\pi^3}{45\beta^4}\frac{(r_-^2+a^2)^3}{(r_+-r_-)^2}\int_{r_--\varepsilon}^{r_--\varepsilon-\delta}\frac{dr}{(r-r_-)^2} \tag{19}$$

$$=\frac{2\pi^3}{45\beta^4}\frac{(r_-^2+a^2)^3}{(r_+-r_-)^2}\frac{\delta}{\varepsilon(\varepsilon+\delta)}. \tag{20}$$

Considering the temperature of the inner horizon $\frac{1}{\beta}=T_-=\frac{r_+-r_-}{4\pi(r_-^2+a^2)}$, the entropy is

$$S_-=\beta^2\frac{\partial F}{\partial\beta}=-\frac{\pi(r_-^2+a^2)}{90\beta}\frac{\delta}{\varepsilon(\varepsilon+\delta)}. \tag{21}$$

Selecting appropriate cut-off ε and δ as $\frac{\delta}{\varepsilon(\varepsilon+\delta)}=90\beta$, the entropy of the inner horizon is

$$S_-=-\frac{1}{4}A_-, \tag{22}$$

where A_- is the area of the inner horizon. The entropy is also proportional to the area of the inner horizon and cut off is 90β which is same in the calculation of the entropy of the outer horizon. The reason why the entropy of the inner horizon is negative is not clear. It is an open question. According to the familiar formula, $T_-=(\frac{dS_-}{dm})^{-1}$, the temperature or the entropy of the inner horizon is negative and another is positive. The entropy was positive and the temperature was negative in several papers before. However, they did not explain clearly why the temperature of the inner horizon is negative. In fact, our understanding of the essence of the entropy of the black hole is still incomplete. Nevertheless, the negative entropy of the inner horizon can help us to let entropy of the black hole with two horizons satisfy the Nernst theorem.

The Nernst theorem claims that the entropy of a system must go to zero as its temperature reaches zero. The entropy and the temperature of the K-N black hole are, $S_{BH} = \pi(r_+^2 + a^2)$ and $T = \frac{\kappa_+}{2\pi}$, respectively. It is obvious that the entropy does not vanish when the temperature, $T \to 0$. If the total entropy of the black hole can be regarded as the sum of the contribution of the outer and inner horizon [13]:

$$S_{BH} = S_+ + S_- = \pi(r_+^2 + a^2) - \pi(r_-^2 + a^2) = \pi(r_+^2 - r_-^2), \tag{23}$$

it is manifest that when the temperature, $T = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}$, goes to absolute zero, $r_+ = r_-$, the entropy of the black hole (23) vanishes. Consequently, the Nernst theorem is satisfied.

4 Bekenstein-Smarr Formula Using the Parameters of the Inner Horizon

$$2\Omega_+ J + V_{0+} Q = r_- = M - \sqrt{M^2 - a^2 - Q^2}, \tag{24}$$

$$\frac{1}{4\pi}\kappa_+ A_+ = \sqrt{M^2 - a^2 - Q^2}. \tag{25}$$

Adding the above two equations, we can obtain the Bekenstein-Smarr integral formula using the parameter of the outer horizon [13]

$$M = \frac{1}{4\pi}\kappa_+ A_+ + 2\Omega_+ J + V_{0+} Q, \tag{26}$$

where $V_{0+} = \frac{Q r_+}{r_+^2 + a^2}$. Differentiating the above equation, we have

$$\delta M = \frac{1}{4\pi}\kappa_+ \delta A_+ + \frac{1}{4\pi}(\delta\kappa_+) A_+ + 2\Omega_+ \delta J + 2(\delta\Omega_+) J + V_{0+}\delta Q + (\delta V_{0+}) Q. \tag{27}$$

After calculation, the differential equation of Bekenstein-Smarr formula is

$$\delta M = \frac{1}{8\pi}\kappa_+ \delta A_+ + \Omega_+ \delta J + V_{0+}\delta Q. \tag{28}$$

Similarly, we can get the following equations adopting the parameters of the inner horizon

$$2\Omega_- J + V_{0-} Q = r_+ = M + \sqrt{M^2 - a^2 - Q^2}, \tag{29}$$

$$\frac{1}{4\pi}\kappa_- A_- = -\sqrt{M^2 - a^2 - Q^2}, \tag{30}$$

where $V_{0-} = \frac{Q r_-}{r_-^2 + a^2}$. Thus, we can rewrite the new Bekenstein-Smarr formula as the parameters of the inner horizon, κ_-, A_-, Ω_- and V_{0-},

$$M = \frac{1}{4\pi}\kappa_- A_- + 2\Omega_- J + V_{0-} Q. \tag{31}$$

In order to get the new differential formula, we differentiate the (31) and have

$$\delta M = \frac{1}{4\pi}\kappa_- \delta A_- + \frac{1}{4\pi}(\delta\kappa_-) A_- + 2\Omega_- \delta J + 2(\delta\Omega_-) J + V_{0-}\delta Q + (\delta V_{0-}) Q. \tag{32}$$

We can work out

$$\frac{1}{4\pi}(\delta\kappa_-)A_- + 2(\delta\Omega_-)J + (\delta V_{0-})Q = -\delta M + V_{0-}\delta Q. \tag{33}$$

We obtain the Bekenstein-Smarr differential formula using the parameters of the inner horizon

$$\delta M = \frac{1}{8\pi}\kappa_-\delta A_- + \Omega_-\delta J + V_{0-}\delta Q. \tag{34}$$

Substituting the temperature and the entropy of the inner horizon which are calculated in Sects. 2 and 3, $T_- = -\frac{\kappa_-}{2\pi}$ and $S_- = -\frac{1}{4}A_-$, we can get

$$\delta M = T_-\delta S_- + \Omega_-\delta J + V_{0-}\delta Q, \tag{35}$$

where the values of T_-, S_-, Ω_-, V_{0-} are on the inner horizon. So, the first law of black hole thermodynamics is tenable.

5 Tunneling Effect of Two Horizons

Because the entropy of the black hole is contributed not only by the outer horizon but also by the inner horizon, the tunneling effect of the inner horizon must be considered. Otherwise, the information will not be conserved. The total tunneling rate should be the product of the tunneling rates of the outer horizon and the inner horizon.

It is manifest that there is a coordinate singularity in the line elements (2) at the horizon. The line element in new coordinates–Painlevè coordinates is [17]

$$\begin{aligned} ds^2 = &-\frac{\rho^2\Delta}{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}dt^2 + 2\sqrt{\frac{\rho^2(\rho^2-\Delta)}{(r^2+a^2)^2-\Delta a^2\sin^2\theta}}dtdr + dr^2 \\ &+ g'_{22}d\theta^2 + 2g'_{12}drd\theta + 2g'_{02}dtd\theta. \end{aligned} \tag{36}$$

In fact, we do not need know the exact form of $g'_{22}, 2g'_{12}$ and $2g'_{02}$. It is obvious that the components are not diverge at the outer and the inner horizon. The radial null geodesics obey

$$\dot{r} = \frac{dr}{dt} = \left[\frac{-\sqrt{\rho^2(\rho^2-\Delta)}+\rho^2}{\sqrt{(r^2+a^2)^2-\Delta a^2\sin^2\theta}}\right] \tag{37}$$

Equations (36) and (37) are modified when the self-gravitation of the particle is considered [14]. We fix the total mass (ADM mass) and allow the hole mass to fluctuate. When the article of energy ω travels on the geodesics, we should replace m with $m-\omega$ in the geodesic (37) and in the line elements (36).

In the semiclassical limit, we can apply the WKB formula. This relates the tunneling amplitude to the imaginary part of the particle action at stationary phase. The emission rate, Γ, is the square of the tunneling amplitude [15, 25]:

$$\Gamma \sim \exp(-2\mathrm{Im}\, I), \tag{38}$$

where I is the action. A positive energy particle created near the singularity travels forward in time. When it arrives at the inner horizon, the tunneling rate of the inner horizon is

$$\Gamma_- \sim \exp(-2\mathrm{Im}\, I_-). \tag{39}$$

The radius of the inner horizon increases when the mass of the black hole decreases. It is this expansion that sets the barrier. Because the energy and the angular momentum of the K-N black hole have, $J = ma$, the angular momentum must change when the energy changes. So, the change both of energy and angular momentum can influence the action I. The imaginary part of the action for an outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} can be expressed as [17, 21]

$$\operatorname{Im} I_- = \operatorname{Im}\left[\int_{r_{in}}^{r_{out}}\int_0^{p_r} dp_r' dr - \int_{\varphi_{in}}^{\varphi_{out}}\int_0^{p_\varphi} dp_\varphi' d\varphi\right], \tag{40}$$

where P_r and P_φ are two canonical momentum which conjugate to r and φ respectively. $r_{in} = r_-$ is the initial radius of the inner horizon, and $r_{out} = r_-'$ is the final radius of the inner horizon, where $r_-' = r_-(m - \omega)$. Here it is noted that $r_-' > r_-$. Applying the Hamilton's equation, we get

$$\dot{r} = \frac{dH}{dP_r} = -\frac{d\omega}{dP_r}, \tag{41}$$

$$\dot{\varphi} = \frac{dH}{dP_\varphi} = \frac{\Omega_- dJ}{dP_\varphi} = -a\Omega_- \frac{d\omega}{dP_\varphi}. \tag{42}$$

We have

$$\operatorname{Im} I_- = \operatorname{Im}\left[\int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}'} d(-\omega') - \int_0^{\omega}\int_{\varphi_{in}}^{\varphi_{out}} \frac{d\varphi}{\dot{\varphi}'} a\Omega_-' d(-\omega')\right]. \tag{43}$$

From $\dot{\varphi} = \frac{d\varphi}{dt}$, $\dot{r} = \frac{dr}{dt}$, we have $\frac{d\varphi}{\dot{\varphi}} = \frac{dr}{\dot{r}}$. At last, (43) is

$$\operatorname{Im} I_- = \operatorname{Im}\left[\int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}'} d(-\omega') - \int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{dr}{\dot{r}'} a\Omega_-' d(-\omega')\right], \tag{44}$$

where $\dot{r}' = \dot{r}(m - \omega')$, $\Omega_-' = \Omega_-(m - \omega')$. Substituting Kerr metric into (44), we obtain

$$\operatorname{Im} I_- = \operatorname{Im}\left[\int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{\sqrt{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}}{\rho^2 - \sqrt{\rho^2(\rho^2 - \Delta)}} \cdot (1 - a\Omega_-') dr(-d\omega'). \tag{45}$$

We have used the modified (37). In (45), $r = r_-'$ is the first order pole. Do the integral r first, we obtain

$$\operatorname{Im} I_- = 2\pi \int_0^{\omega} \frac{r_-'^2}{r_+' - r_-'} d\omega'. \tag{46}$$

So, the tunneling rate is

$$\Gamma_- = \exp[-2\operatorname{Im} I_-] = \exp\left[-\int_0^{\omega} 4\pi \left(\frac{r_-'^2}{r_+' - r_-'}\right) d\omega'\right]. \tag{47}$$

Using the same method, the tunneling rate of the outer horizon is

$$\Gamma_+ = \exp\left[-\int_0^{\omega} 4\pi \left(\frac{r_+'^2}{r_+' - r_-'}\right) d\omega'\right]. \tag{48}$$

Thus the total tunneling rate is

$$\Gamma = \Gamma_+ \cdot \Gamma_- = \exp\left[-\int_0^{\omega} 4\pi\left(\frac{r_+'^2 + r_-'^2}{r_+' - r_-'}\right)d\omega'\right]. \tag{49}$$

Though it is difficult to work out the integral with respect to ω' directly, we can make the physical meaning clear as follows. The entropy (23) derivative of m is

$$\begin{aligned}\frac{\partial S_{BH}}{\partial m} &= \pi\left(2r_+\frac{\partial r_+}{\partial m} - 2r_-\frac{\partial r_-}{\partial m}\right)\\ &= 4\pi\left(\frac{r_+^2 + r_-^2}{r_+ - r_-}\right).\end{aligned} \tag{50}$$

Doing the integral of m, (50) becomes to

$$\Delta S_{BH} = \int_m^{m-\omega}\frac{\partial S_{BH}}{\partial m'}dm' = \int_m^{m-\omega} 4\pi\left(\frac{r_+'^2 + r_-'^2}{r_+' - r_-'}\right)dm'. \tag{51}$$

Substituting $m' = m - \omega$ into above equation, we have

$$\Delta S_{BH} = -\int_0^{\omega}\frac{\partial S_{BH}}{\partial m'}d\omega' = -\int_0^{\omega} 4\pi\left(\frac{r_+'^2 + r_-'^2}{r_+' - r_-'}\right)d\omega'. \tag{52}$$

Comparing (49) with (52), we find

$$\Gamma \to \exp(\Delta S_{BH}). \tag{53}$$

This result is in agreement with Parikh' work.

6 Discussion and Conclusion

We calculate the temperature of the inner horizon of the K-N black hole and show that the inner horizon has thermal characters. The quantum effect near the inner horizon can be called "Hawking absorption". If the contribution of the inner horizon is taken into account to the entropy of the black hole, the Nernst theorem can be satisfied. Hence, we suggest that the black hole with two horizons is regarded as a thermodynamics system composed of two subsystem, the outer horizon and inner horizon. Our work of the radiation effect of the inner horizon has much importance because it supports the idea that all horizons of space-time emit radiation.

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